David Mumford

Archive for Reprints, Notes, Talks, and Blog

Professor Emeritus
Brown and Harvard Universities
David_Mumford@brown.edu

Ridiculous Math Problems

April 1, 2020

Maybe some humor is useful while we are all caught up in the depressing whirlwind of this Covid-19 pandemic. Recently, a silly math problem went viral on facebook about what appear to be prices for a doll, a pair of shoes and a pair of scarves (reproduced below). It is an example of how the public loves brain-teasers, odd puzzles with some math in them. What makes this interesting to me is that this playing with really meaningless math problems is something that mathematicians do too. There is an ancient tradition of ridiculous math problems that permeate the history of math, especially the history of algebra. I find it odd that no book on the History of Math points out how many algebra problems in every era are crazy concoctions whose main point is to show how smart their creator was or how nifty their discoverer's new tool is. It's a fascinating, not well-known side of math.

Curiously, the creation of silly math problems goes back to the earliest known mathematical documents. A truly ridiculous question was posed four thousand years ago on a Babylonian tablet inscribed with cuneiform and concerns solving for numbers involved with building a wall. You are given the *sum* of the number of laborers, number of days needed and the number of loads of bricks. Of course, such a sum has no significance whatsoever and no administrator would ever need to solve any problem like this. Never mind: the problem was probably devised to test the poor student's knowledge of the quadratic formula. Or could it have been a brain-teaser for scribes in their leisure time? Here's what the actual cuneiform says:

> *I added the bricks, the laborers and the days so that it was 140. The days were 2/3rd's of my workers.* (It was also assumed known that a worker can carry 9 / 60th of a load each day). *Select the bricks, the laborers and the days for me.*

If you figure out that there were 30 laborers, 20 days and 90 loads of bricks, you are get a gold star.

This problem sounds like a lot of the so-called "word problems", notorious as the hardest part of high school algebra. It reminds me of the chestnut: "If Jim can dig this ditch in 2 days and Bob can dig it in 3 days, how long would it take them if they dig together?" Actually, I think that problem a pretty good one to master. It requires the student to realize that Jim digs 1/2 the ditch in one day, Bob 1/3 of the ditch, because the number of days and the fraction dug in one day are inverses of each other. I leave it to the reader to finish it (a silver star here).

A great example of a truly ridiculous problem comes from an autobiographical piece by Richard Feynman describing his work reviewing textbooks for the California Board of Education:

> Finally I come to a book that says, "Mathematics is used in science in many ways. We will give you an example from astronomy, which is the science of stars." I turn the page, and it says, "Red stars have a temperature of four thousand degrees, yellow stars have a temperature of five thousand degrees . . ." – so far, so good. It continues: "Green stars have a temperature of seven thousand degrees, blue stars have a temperature of ten thousand degrees, and violet stars have a temperature of . . . (some big number)." There are no green or violet stars, but the figures for the others are roughly correct. It's vaguely right – but already, trouble! ....
>
> Anyway, I'm happy with this book, because it's the first example of applying arithmetic to science. I'm a bit unhappy when I read about the stars' temperatures, but I'm not very unhappy because it's more or less right – it's just an example of error. Then comes the list of problems. It says, "John and his father go out to look at the stars. John sees two blue stars and a red star. His father sees a green star, a violet star, and two yellow stars. What is the total temperature of the stars seen by John and his father?" – and I would

> explode in horror. My wife would talk about the volcano downstairs.

It's always makes me laugh -- adding temperatures of some set of objects is such a nutty meaningless idea. Even in this pandemic, it would be absurd if a hospital were to post the total temperature of all its patients!

Curiously, every advance in algebra in every culture appears, as far as I know, with the appearance of such useless but always tricky puzzles. Ancient Greek math was not known for its algebra wi†h the exception of the work of Diophantus. What do his problems look like? Here's a typical one:

> IV.39: To find three numbers such that the difference of the greatest and the middle has to the difference of the middle and the least a given ratio, and further such that the sum of any two is a square.

What is always implicit in his book is that he wants all his numbers to be positive rational fractions. In this specific case, he specializes the problem to ask for the ratio to be 3 and comes up with expressions for the three numbers depending on a fourth rational number that you can choose as any fraction between 0 and 2. In this case, he gives this representative answer: 29/242, 939/242 and 3669/242! He had a few tricks for coming up with such bizarre numbers and he spun this out to hundreds of such problems.

But despite its apparently pointless

appearance, Diophantus's problems have been very fruitful -- in pure mathematics. One the greatest 20th century mathematicians, André Weil, wrote a book in which he analyzed Diophantus using contemporary math, algebraic geometry and number theory, to reveal the logic behind his choice of problems. Today, the study of integer and rational solutions of polynomial equations is know as "Diophantine Analysis".

Let me make a short list chronicling how every advance in algebra was accompanied by ridiculous problems:

1. **Bhaskara II, Ujjain, India 1114-1185 CE** Bhaskara II developed the algebra of Brahmagupta, the father of algebra in India, whose writings apparently made their way to the caliphate in Baghdad and likely inspired al-Khwarizmi. Like many others, he could not resist the temptation to show how powerful were his ideas with a meaningless problem:

   > *If thou be conversant with operations of algebra, tell the number of which the biquadrate* (4th power) *less double the sum of the square and 400 times the simple number is a myriad* (10,000) *less one." (Vija-Ganita, V.138)*

   Well this is a bizarre 4th degree polynomial equation, $x^4 - 2x^2 - 400x = 9999$ and he applies one approach to solving it, then comes to a dead end and says "Hence ingenuity is called for" and finally finds that his unknown number is 11. I, personally, would not have had a clue how to solve it.

2. **Fibonacci, Pisa c.1170-c.1240 CE** Son of a world trader who took him to Africa and Asia, Leonardo de Pisa (his proper name) wrote a remarkable book *Liber Abaci* that introduced Europe not only to arabic numerals but also to algebra and its rules. After chapters on the basics, the book is mostly a huge collection of concocted problems of which I want to give an example belonging to a class of traditional but unrealistic money puzzles:

> Three men had pounds of sterling, I know not how many, of which one half was the first's, one third was the second's and one sixth's was the thirds; as they wished to have it in a place of security, every one of them took from the sterling some amount, and of the amount that the first took he put in common one half, and of it that the second took, he put in common a third part, and of that which the third took, he put in common a sixth part, and from that which they put in common every one received a third part, and thus each had his portion.

$$\frac{x_1}{2}+\frac{1}{3}\left(\frac{x_1}{2}+\frac{x_2}{3}+\frac{x_3}{6}\right)=\frac{1}{2}(x_1+x_2+x_3)$$
$$\frac{2x_2}{3}+\frac{1}{3}\left(\frac{x_1}{2}+\frac{x_2}{3}+\frac{x_3}{6}\right)=\frac{1}{3}(x_1+x_2+x_3)$$
$$\frac{5x_3}{6}+\frac{1}{3}\left(\frac{x_1}{2}+\frac{x_2}{3}+\frac{x_3}{6}\right)=\frac{1}{6}(x_1+x_2+x_3)$$

This is 'just' a simple set of three linear equation in three unknowns. But even with modern methods, I struggled not to make arithmetic mistakes solving them.

Gold star if you find 33:13:1 for wealth of the three men

3. **Tartaglia, Venice, 1535** A most extraordinary competition occurred in Northern Italy in the first half of the sixteenth century *over formulas for solving polynomial equations of degree 3 and 4*! From the time of the Babylonians, it was known how to solve quadratic equations. Why the problem of higher degrees obsessed Italians is unknown, at least to me, but the story apparently started with one Scipione del Ferro in Bologna discovering the formula for one type of third degree polynomials early in the sixteenth century *but keeping the rule a secret!* However, he told the rule to his student Antonio Fior. Meanwhile, Niccolo Tartaglia found a formula for another type of cubic and challenged Fior, not to the customary duel, but to solve 30 cubic equations that each sent to the other! During the night of Feb. 12-13, 1535, Tartaglia had an inspiration and rapidly solved all of Fior's equations. The story continues: Gerolamo Cardano inveigles the formula out of Tartaglia and then, with the help of Ludovico Ferrari, worked out the formula for 4th degree polynomials as well. When, against his sworn word, Cardano published both, Tartaglia was incensed and challenged them to a debate in Milan, 1548. He lost, sued, lost again and retreated in disgrace to Venice. Seldom has math led to such public clashes. Cardano, however, had published his book, *Ars Magna*, that immortalized

all this joint work. Here is an excerpt from this sixteenth century best seller:

> For example,
>
> $$x^3 + 6x = 20.$$
>
> Cube 2, one-third of 6, making 8; square 10, one-half the constant; 100 results. Add 100 and 8, making 108, the square root of which is $\sqrt{108}$. This you will duplicate: to one add 10, one-half the constant, and from the other subtract the same. Thus you will obtain the *binomium* $\sqrt{108} + 10$ and its *apotome* $\sqrt{108} - 10$. Take the cube roots of these. Subtract [the cube root of the] *apotome* from that of the *binomium* and you will have the value of x:
>
> $$\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$$

He chooses a cubic equation with coefficients 6 and 10, apparently more or less at random to show how his formula works. (The odd thing is, he doesn't mention that these specific cube roots can be evaluated and equal $\sqrt{3} \pm 1$, so that $x = 2$ is the solution. Of course, this is particular to his choice of coefficients.) Can you imagine a crowd turning up today to hear two math guys argue over solving oddball equations? But I should add: like Diophantus, Cardano's work started a major area in pure math called 'Galois theory'.

Note that there are no Chinese mathematicians in this list. This is not an accident. Essentially all Chinese scholars (that is, the Mandarins) had no interest in mathematics and math was left to those lower on the pecking order who did not indulge in playing with irrelevant problems, focussing entirely on practical math. Their basic text was the "Nine Chapters on the Mathematical Art", put together in the Han dynasty (206 BCE -- 220 CE). The book is almost entirely examples, giving practical rules for various problems encountered administering the empire. Here is one concerning the price of rice of varying qualities:

Now given 3 bundles top grade paddy, 2

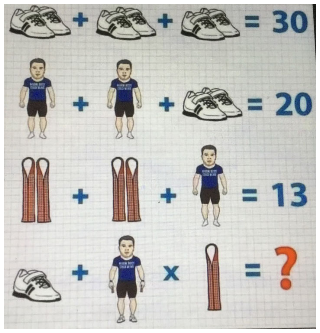
> bundles medium grade, 1 bundle low grade. Yield: 39 dou of grain. 2 bundles top, 3 bundles medium, 1 bundle low. Yield 34 dou. 1 bundle top, 2 bundles medium, 3 bundles low. Yield 26 dou. Tell: how much paddy does one bundle of each grade yield?

This is exactly the same math as in Fibonacci's problem and is awfully similar to the viral problem discussed next. The Chinese solved it using sticks laid out on a grid and moved rapidly following some basic rules, but, in standard math terms, it translates to:

$$\begin{aligned} 3T + 2M + 1L &= 39 \\ 2T + 3M + 1L &= 34 \\ 1T + 2M + 3L &= 26 \end{aligned}$$

Here T is the price of top grade, M medium and L low. Just as in Fibonacci, 3 linear equations in 3 unknowns, but now the Chinese had discovered the optimal way to solve them. It is called "Gaussian elimination" in the West but should be called "Chinese elimination". Fibonacci's method is very obscure. A star for solving this: I won't give the answer this time.

OK, let me get to the nutty recently viral problem is given by this image:



An array of equations, like the last one or Fibonacci's problem. What do the numbers on the right mean? You can see both

The Puzzle	single shoes and pairs. Are the objects unknowns? If so, what do you make of the doll on the last line wearing shoes? Is this a new unknown or an addition without a plus sign?

The only meaning I can see is that the number are prices, e.g. from the first equation, a pair of shoes is worth $10; from the second, the man doll is worth $5; and from the third, a pair of scarves is worth $4. It's a bit like the Babylonian problem where numbers of workers, days, bricks were added. But a twist -- in the last equation, a single shoe is added to the *product* of (i) a doll wearing two shoes and carrying two scarves and (ii) a single scarf. Clearly, you're supposed to to disregard the inappropriateness of multiplication here and come up with 5+2*(5+10+4)=43. However, it is meaningless to multiply $2 by $19 and all I can say is that the problem is a screwy brain-teaser. For me as an applied math guy, disregarding the units of measurement when carrying out arithmetic operations (as the Babylonians did in the first example) is one the cardinal sins.

Thanks to Cornelius Mika, I have learned that I got the identity of the objects in this challenging bit of math quite wrong: the "dolls" are actually weightlifters, their shoes are specifically weightlifting shoes and the "scarves" are weightlifting straps. Well, let's get it right.

I need to admit that math puzzles are a lot of fun. There is a whole cult following the puzzles and games that Martin Gardner wrote up in his Scientific American columns. And KenKen is addictive. I grew up with Alcuin's famous wolf and river problem dating from about 1200 years ago:

> A man had to take a wolf, a goat and a bunch of cabbages across a river. The only boat he could find could only take one passenger or baggage at a time. But he had been ordered to transfer all of these to the other side in good condition. How could this be done?

Suffice it to say that the solution requires you to bring various things *back* while ferrying others across. (There's also an X-rated variant with condoms that I hesitate to reproduce.) Let's all pray that this pandemic passes before we run out of ridiculous problems to solve.